CS103
Winter 2026

Lecture 01:

Mathematical Proofs

Mathematical Proofs

1. **Announcements**
2. What is a Proof? (Proof as Dialogue)
3. The Proof Writing Triad
4. Writing Our First Proof
5. Our Second Proof
6. Interlude: More Even/Odd Results
7. Universal and Existential Statements
8. Floors, Ceilings, and Proofs by Cases
9. Recap, Action Items, and What’s Next?

Working in Pairs

- Starting with Problem Set One, you are allowed to work either individually or in pairs.
  - Each pair should make a **single joint submission**.
- We have advice about how to work effectively in pairs up on the course website; check the "Guide to Partners."
- Want to work in a pair, but don't know who to work with? Fill out ***this Google form*** by 5 PM on Friday, and we'll connect you with a partner sometime on Friday evening.

Problem Set 0

- Problem Set 0 is due this ***Friday*** at ***1:00 PM***.
  - It needs to be completed individually.
- Need help getting Qt Creator installed? There’s a Qt Creator help session running ***Friday***, ***2:30 – 4:30 PM***, in ***CoDa B45***.
  - This is an emergency backstop in case you run into insurmountable issues. You can use the form on the course website to request an extension if you can’t submit PS0 by 1:00 PM on Friday because of Qt Creator install issues.
  - You can post on Ed for help, as well.

CS103 ACE

- ***CS103 ACE*** is an optional, one-unit companion course to CS103.
- ACE provides additional practice with the course material in a small group setting.
- Meets Tuesdays, 1:30 – 3:20 PM.
- Interested? Apply online using ***this link.*** Deadline is this Friday, 11:59 PM.

Evelyn Yee
(they/them)
ACE Instructor

Outdoor Activities

- You’re less than fifty miles from grassy mountains, redwood forests, Pacific coastline, beautiful wetlands, and more.
- Want to explore the area to see what it has to offer? Check out our (unofficial) Outdoor Activities Guide.

https://cs103.stanford.edu/outdoor_activities

- A sampler of what to check out:
  - Drive to the observatory in the mountains near San Jose and take in the views.
  - Visit a beach with an enormous colony of elephant seals.
  - Walk in redwood forests and pick your own bay leaves.
  - Grab cheap, high-quality food from unassuming strip malls.

Administrative

Be sure to check out Anisha's announcement on Ed.

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Terms have precise, unambiguous definitions.

Mathematical Proofs

Precise, clearly articulated, logically sound arguments.

Proof as Dialogue

- A mathematical proof is a dialogue between two parties: a ***proof writer*** and a ***proof reader***.
  - The ***proof writer*** knows a mathematical fact.
  - The ***proof reader*** is honest but skeptical.
    - The proof writer's job is to take the reader on a journey from ignorance to understanding.

Proof Writer (You)

Proof Reader

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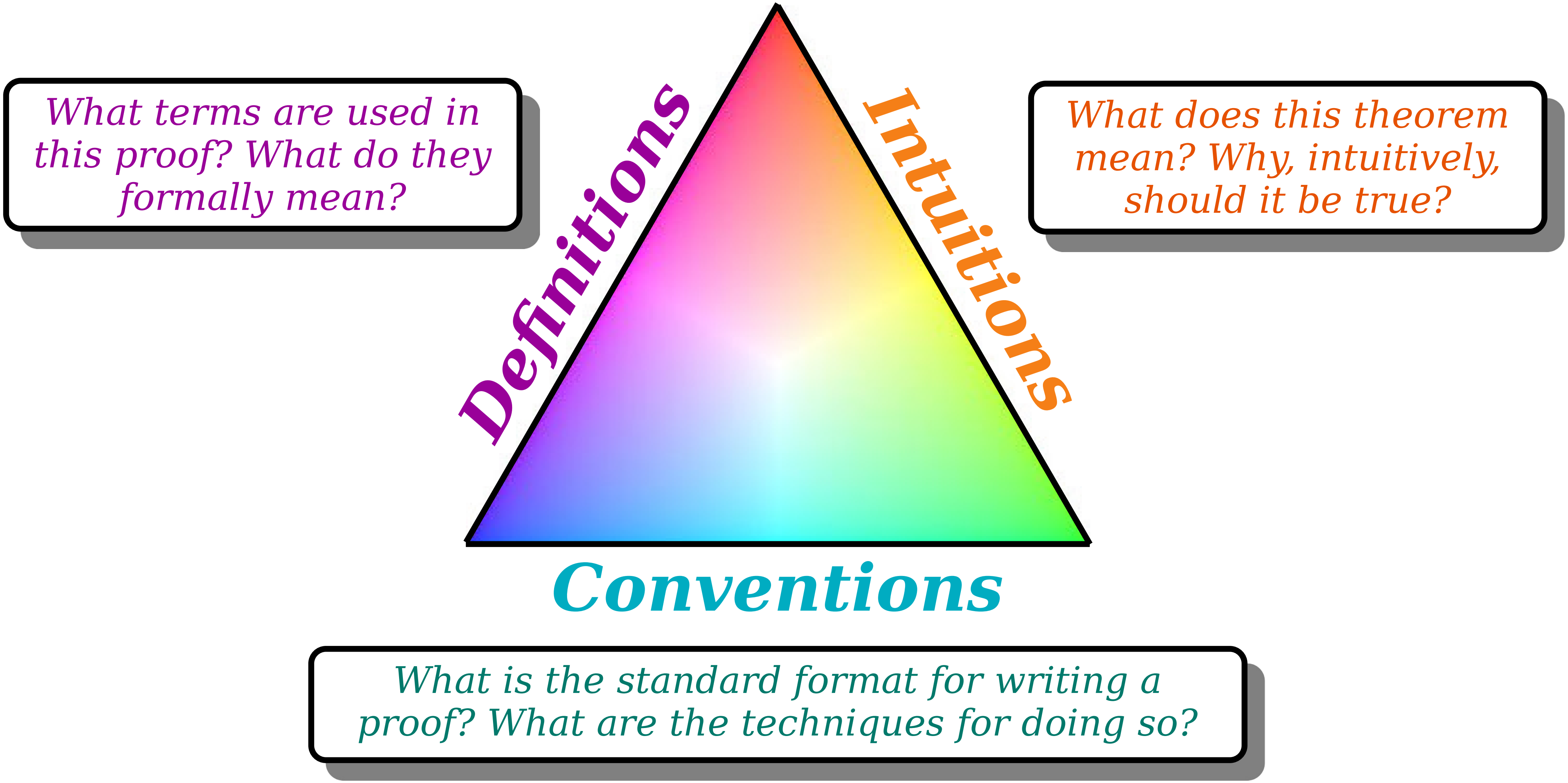
The Proof Writing Triad



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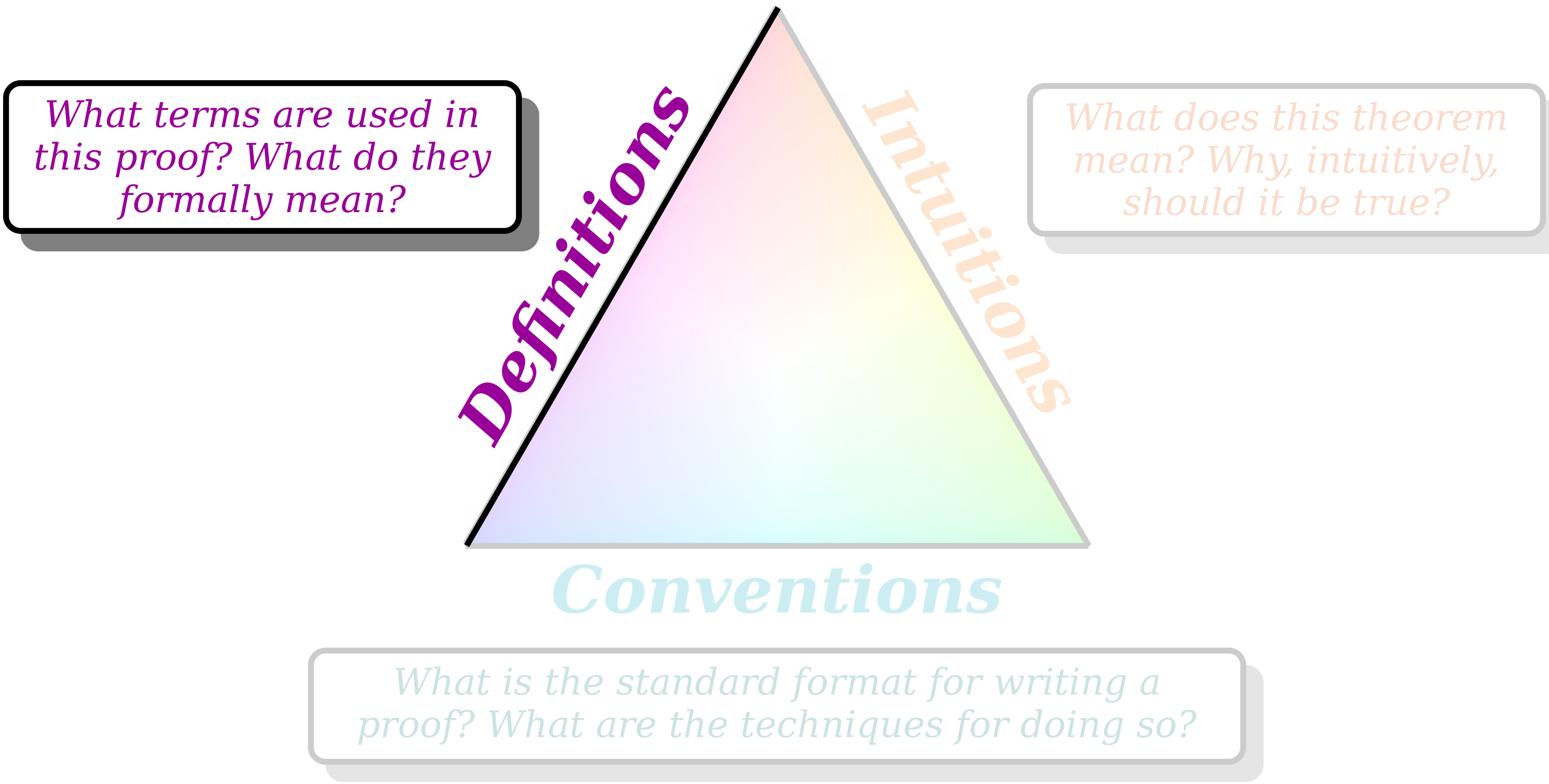
Writing Our First Proof

Theorem: If n is an even integer, then n^2 is even.

Writing Our First Proof

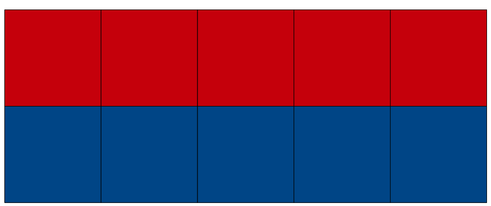


Theorem: If n is an even integer, then n^2 is even.

Writing Our First Proof

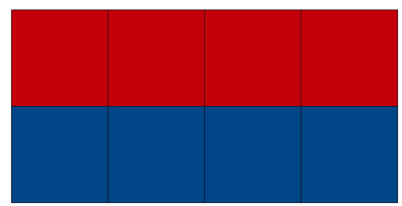
An integer n is called ***even*** if
there is an integer k where $n = 2k$.

10



$2 \cdot \mathbf{5}$

8

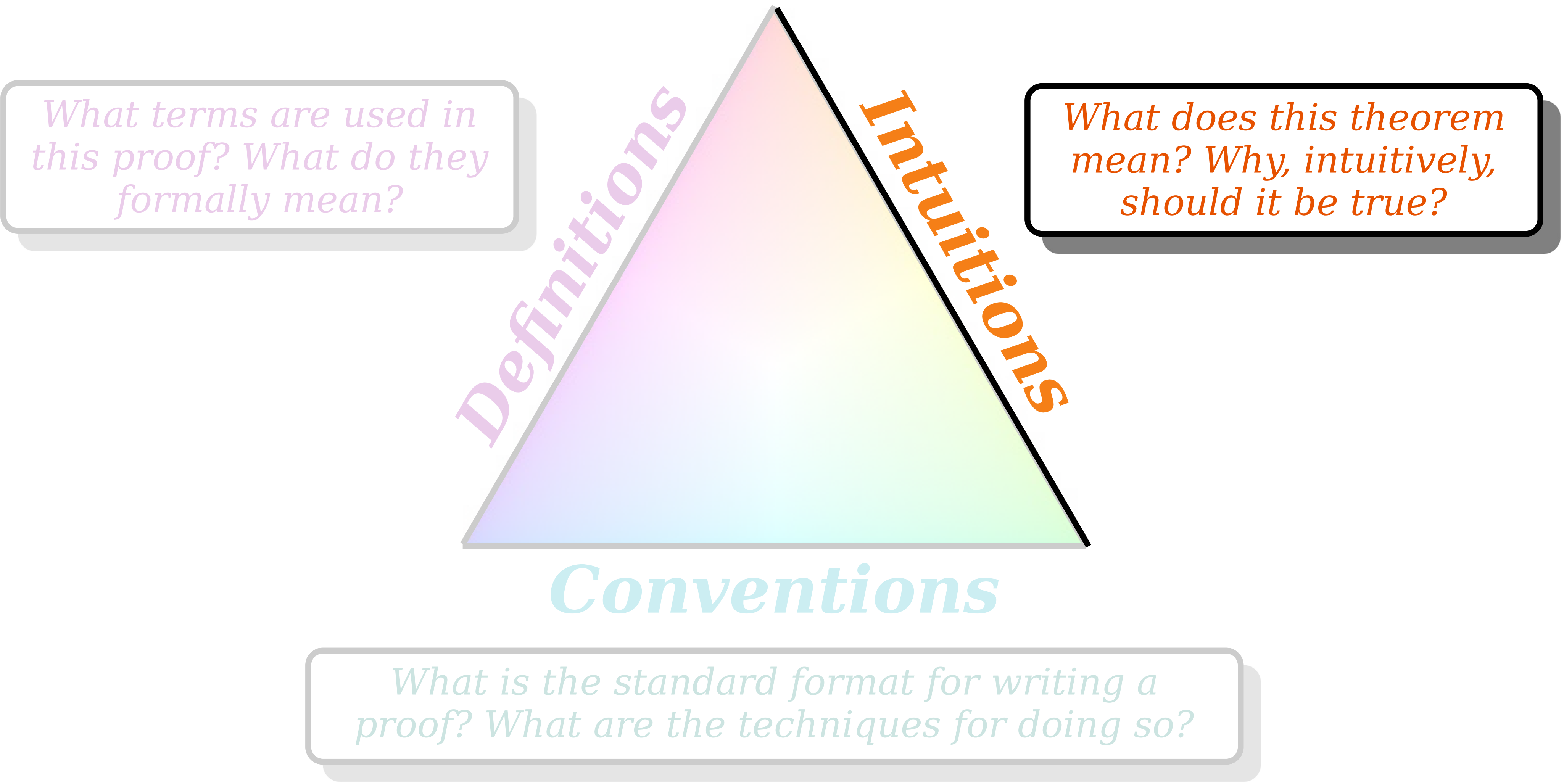


$2 \cdot \mathbf{4}$

0

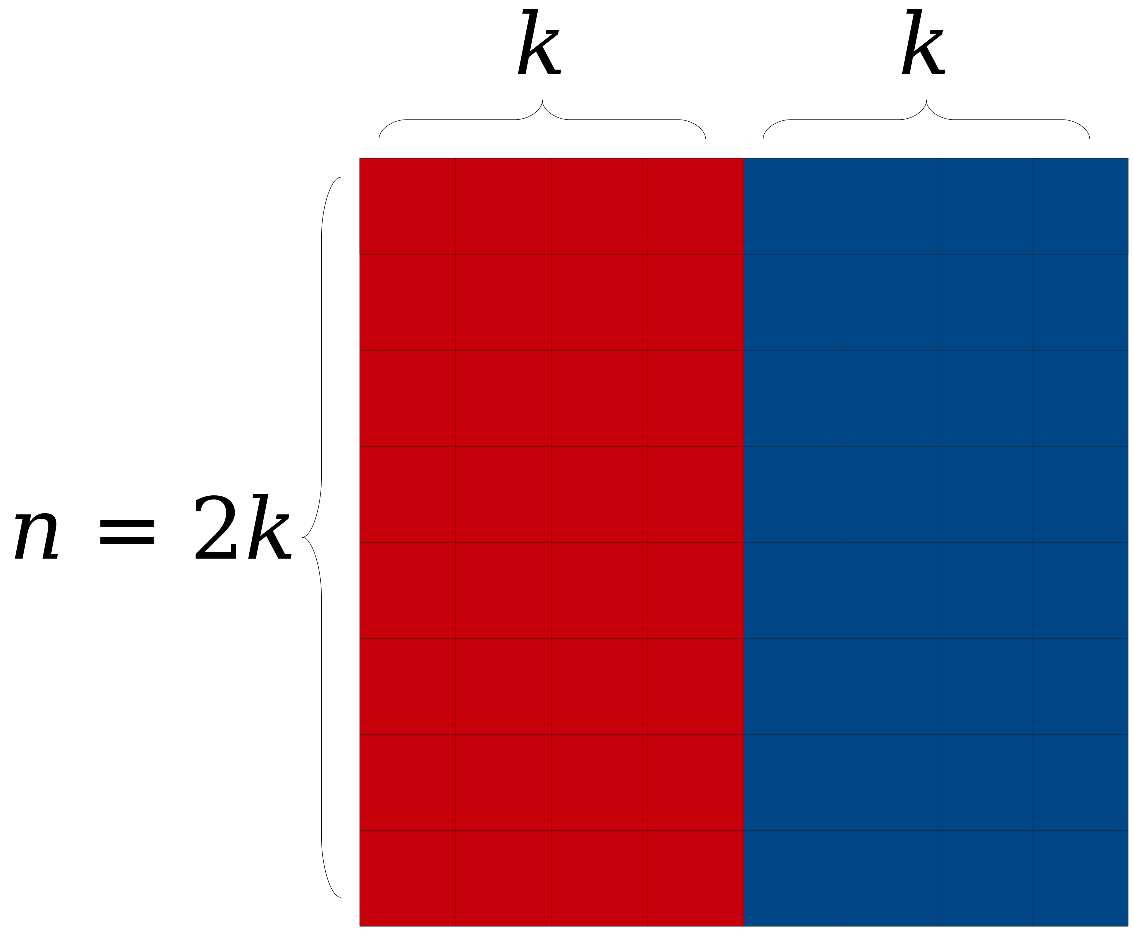
$2 \cdot \mathbf{0}$

Writing Our First Proof



Theorem: If n is an even integer, then n^2 is even.

Let's Draw Some Pictures!

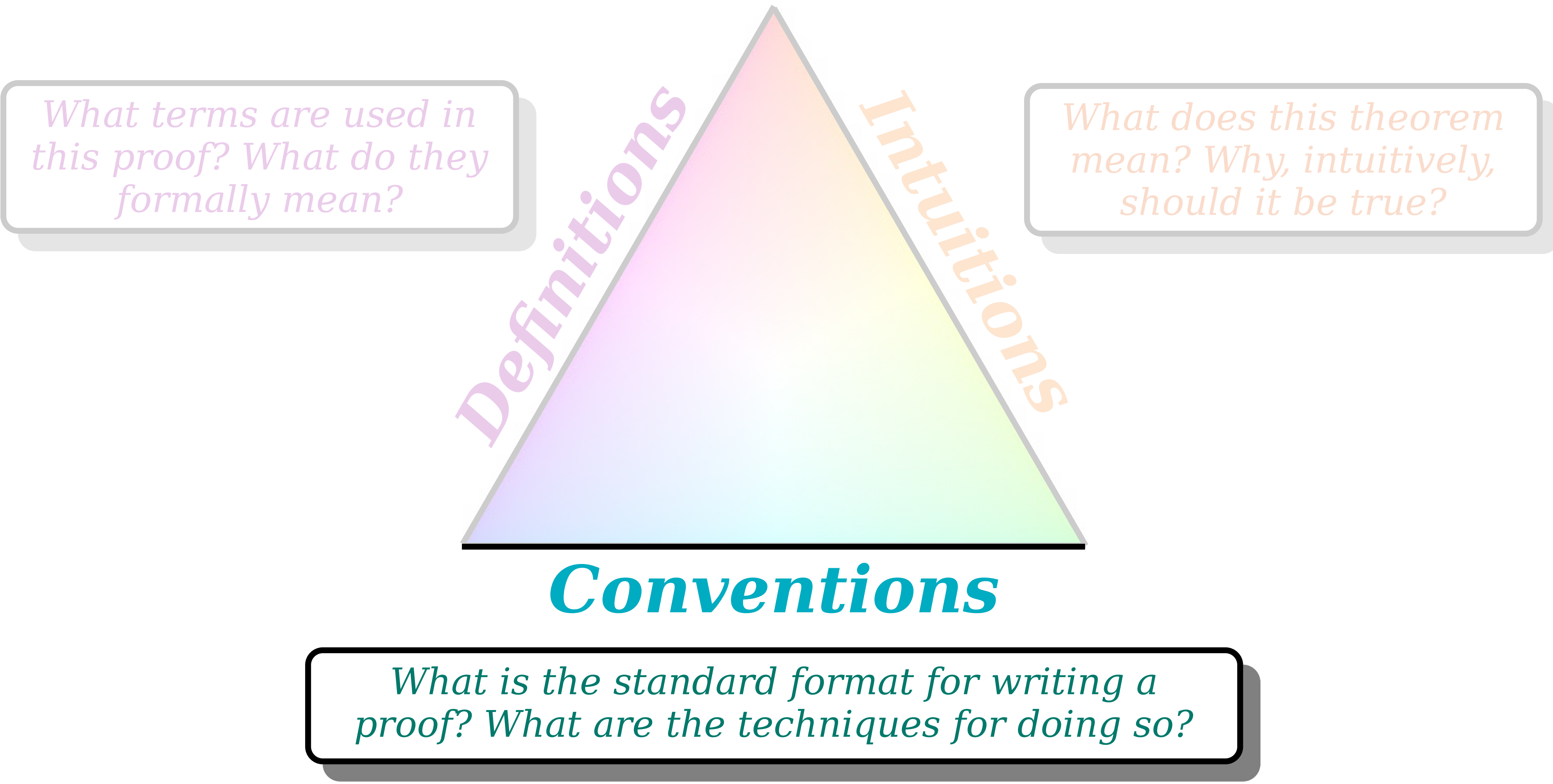


$$n^2 = 2(2k^2)$$

$n = 2k$

Theorem: If n is an even integer, then n^2 is even.

Writing Our First Proof



Theorem: If n is an even integer, then n^2 is even.

Writing Our First Proof

Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$. This means that

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

This symbol means "end of proof"

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

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To prove a statement of the form

"If *P* is true, then *Q* is true,"

start by asking the reader to assume that ***P*** is true.

Writing Our First Proof

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To prove a statement of the form

“If *P* is true, then *Q* is true,”

we assume ***P*** is true, then need to show that ***Q*** is true. Here, we're telling the reader where we're headed.

Writing Our First Proof

Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$. This means that

We apply the definition of an even integer. We need to use this definition to make this proof rigorous.

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

Writing Our First Proof

Theorem: If n

Proof: Assume
that n^2 is even.

Notice how we use the value of ***k*** that we obtained above. Giving names to quantities, allows us to manipulate them. This is similar to variables in programs.

Since n is even, there is some integer k such that $n = 2k$. This means that

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

Writing Our First Proof

Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Since n …
$n = 2k$. …

> Our ultimate goal is to prove that $\boldsymbol{n^2}$ is even. This means that we need to find some $\boldsymbol{m}$ where $\boldsymbol{n^2 = 2m}$. Here, we're explicitly showing how we can do that.

$= 4k^2$
$= 2(2k^2)$.

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

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$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

> Hey, that's what we said we were going to do! We're done.

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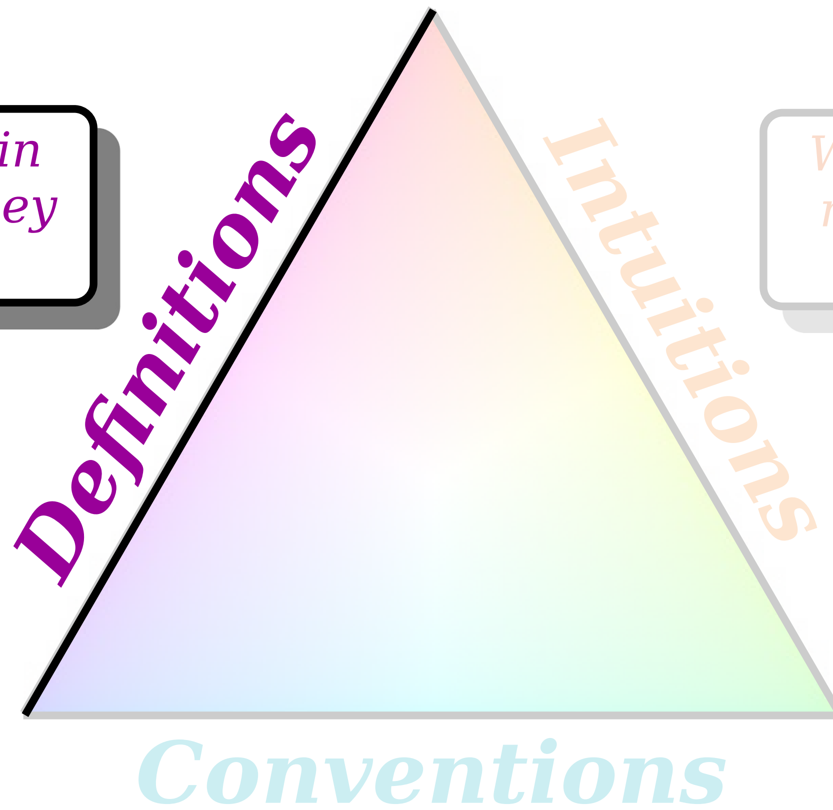
Our Next Proof

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Our Next Proof

What terms are used in this proof? What do they formally mean?



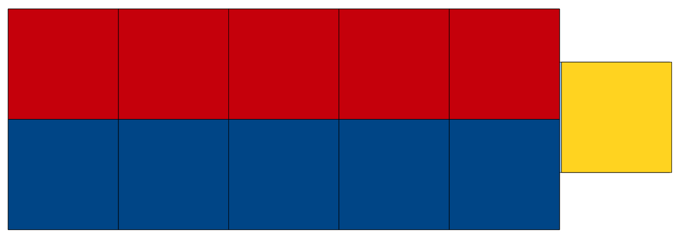
What does this theorem mean? Why, intuitively, should it be true?

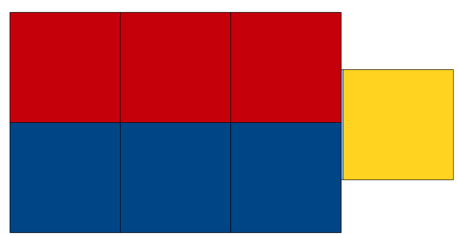
What is the standard format for writing a proof? What are the techniques for doing so?

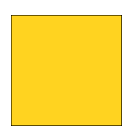
Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Our Next Proof

An integer n is called ***odd*** if
there is an integer k where $n = 2k + 1$.

11  $2 \cdot \mathbf{5} + \mathbf{1}$

7  $2 \cdot \mathbf{3} + \mathbf{1}$

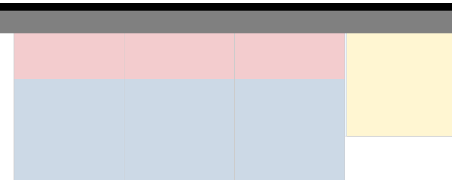
1  $2 \cdot \mathbf{0} + \mathbf{1}$

Our Next Proof

An integer n is called ***odd*** if there is an integer k where $n = 2k + 1$.

Going forward, we'll assume the following:

1. Every integer is either even or odd.
2. No integer is both even and odd.

7  $2 \cdot \mathbf{3} + \mathbf{1}$

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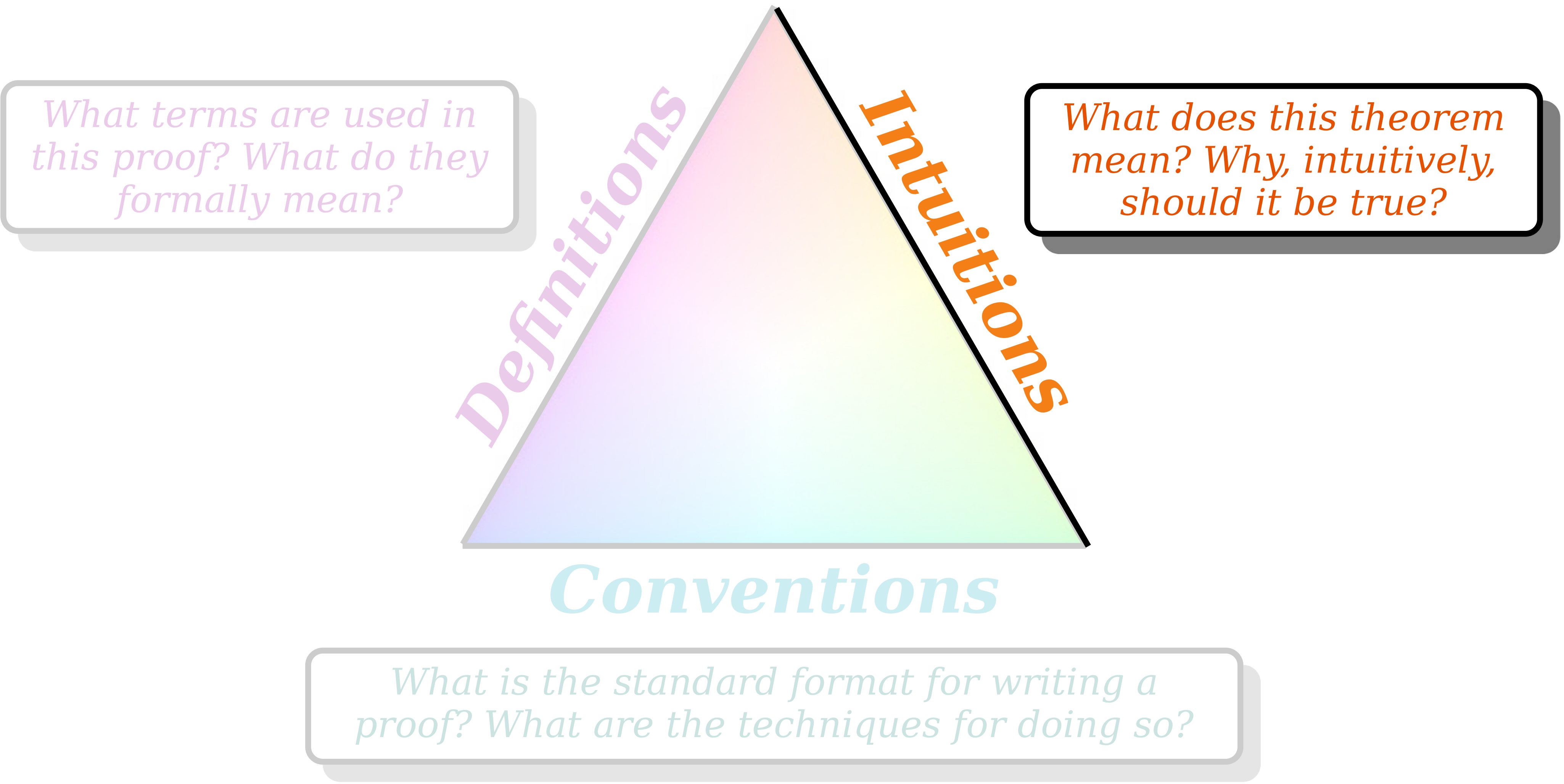
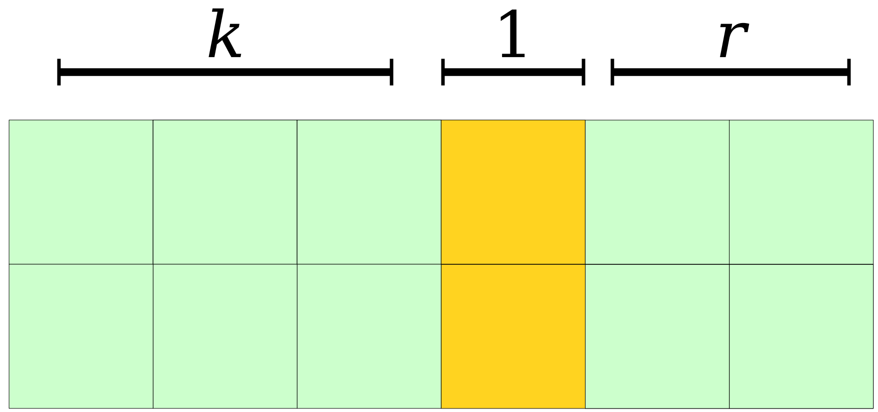


Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Let’s Do Some Math!

$2k+1$

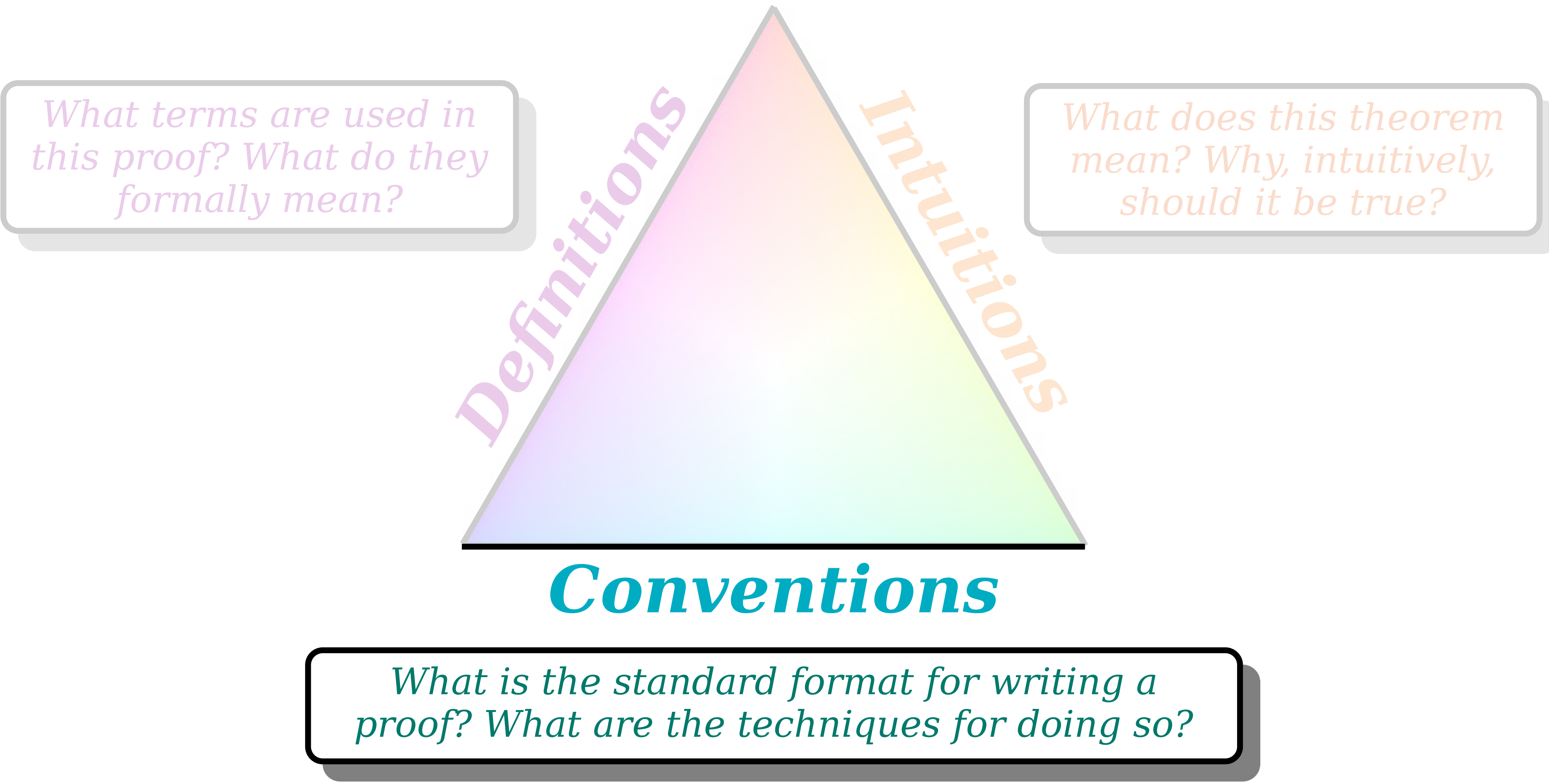


$2r+1$

$$(2k+1) + (2r+1) = 2(k + r + 1)$$

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Our Next Proof



Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

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Proof: Consider any arbitrary integers m and n where m and n are odd. We need to show that $m + n$ is even.

Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \qquad (1)$$

Similarly, because n is odd there must be some integer r such that

$$n = 2r + 1. \qquad (2)$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \qquad (3) \end{aligned}$$

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

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Similarly, becau

By adding equat

Equation (3) tell
such that $m + n$
required. ■

We ask the reader to make an *arbitrary choice*. Rather than specifying what m and n are, we're signaling to the reader that they could, in principle, supply any choices of m and n that they'd like.

By letting the reader pick m and n arbitrarily, anything we prove about m and n will generalize to all possible choices for those values.

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

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Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

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...nteger s (namely, $k + r + 1$)

...see that $m + n$ is even, as

This is a complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

We recommend using the "mugga mugga" test – if you read a proof and replace all the mathematical notation with "mugga mugga," what comes back should be a valid sentence.

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Interlude: More Even/Odd Results

- Here's a list of other theorems that are true about odd and even numbers:
  - ***Theorem:*** The sum and difference of any two even numbers is even.
  - ***Theorem:*** The sum and difference of an odd number and an even number is odd.
  - ***Theorem:*** The product of any integer and an even number is even.
  - ***Theorem:*** The product of any two odd numbers is odd.
- Going forward, we'll just take these results for granted. Feel free to use them in the problem sets.
- If you'd like to practice the techniques from today, try your hand at proving these results!

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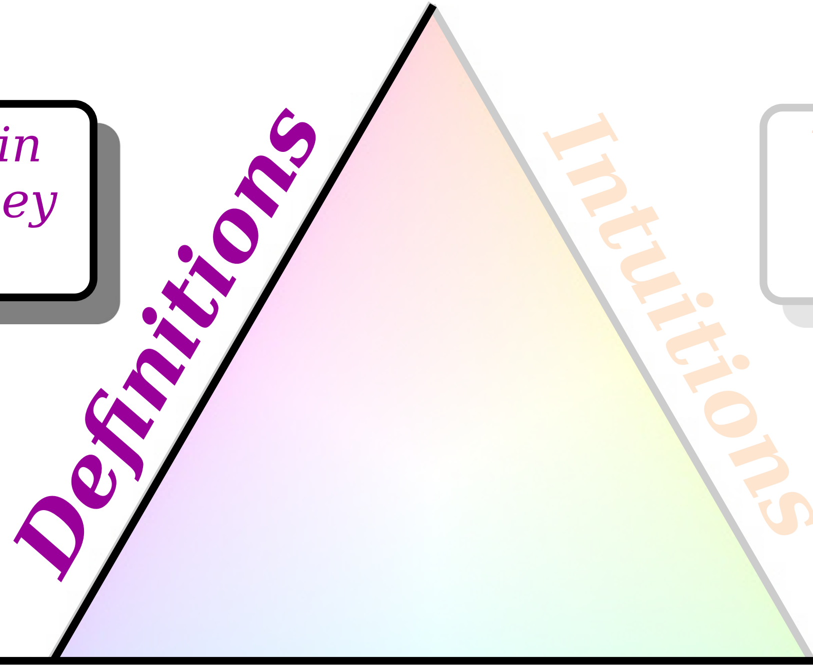
Universal and Existential Statements

Theorem: For any odd integer n,
there exist integers r and s where $r^2 - s^2 = n$.

Universal and Existential Statements

What terms are used in this proof? What do they formally mean?



What does this theorem mean? Why, intuitively, should it be true?

This result is true for every possible choice of odd integer n. It'll work for $n = 1$, $n = 137$, $n = 103$, etc.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Universal and Existential Statements

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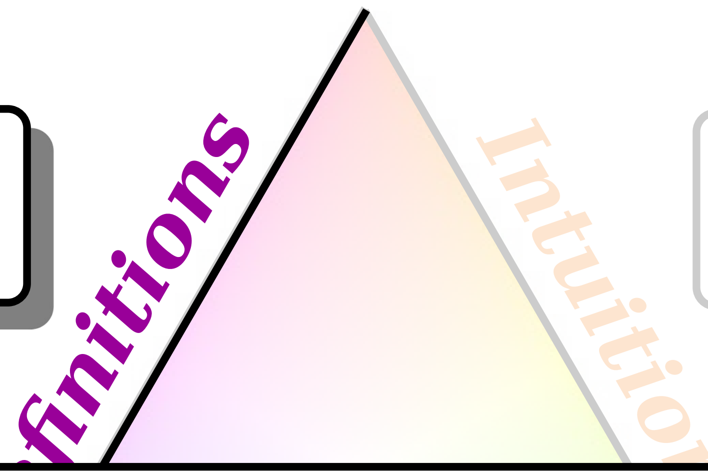


What does this theorem mean? Why, intuitively, should it be true?

We aren't saying this is true for every choice of r and s. Rather, we're saying that *somewhere out there* are choices of r and s where this works.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Universal vs. Existential Statements

- A ***universally-quantified statement*** is a statement of the form, "**For all *x*, `[some-property]` holds for *x*.**"

  We've seen how to prove these statements.

- An ***existentially-quantified statement*** is a statement of the form, "**There is an *x* where `[some-property]` holds for *x*.**"

  How do we prove an existentially quantified statement?

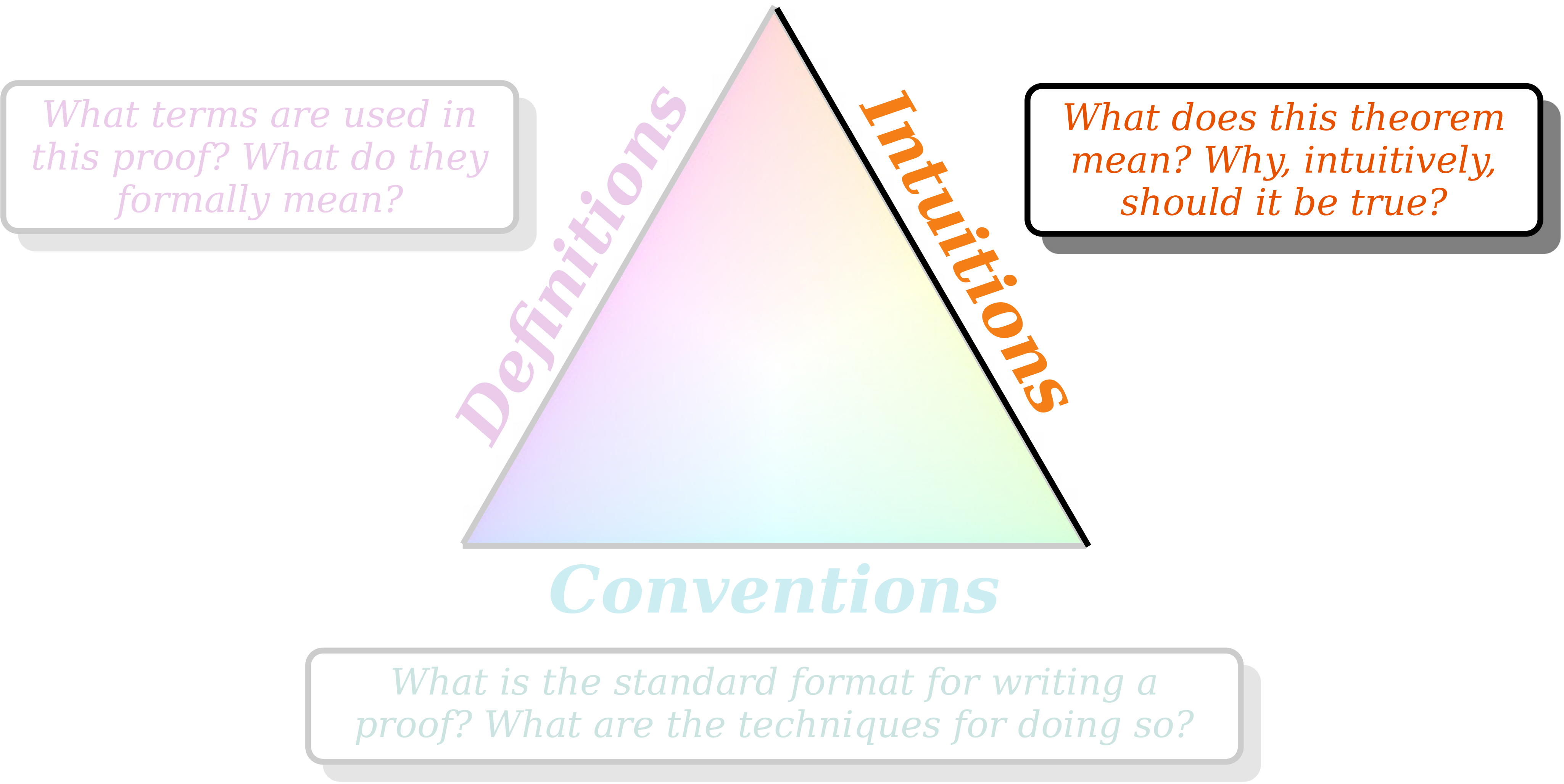
Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existentially-quantified statement of the form

There is an *x* where `[some-property]` holds for *x*.

- ***Simplest approach:*** Search far and wide, find an *x* that has the right property, then show why your choice is correct.

Universal and Existential Statements



Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Let's Try Some Examples!

$$1 = 2 \cdot \mathbf{0} + 1 = \mathbf{1}^2 - \mathbf{0}^2$$

$$3 = 2 \cdot \mathbf{1} + 1 = \mathbf{2}^2 - \mathbf{1}^2$$

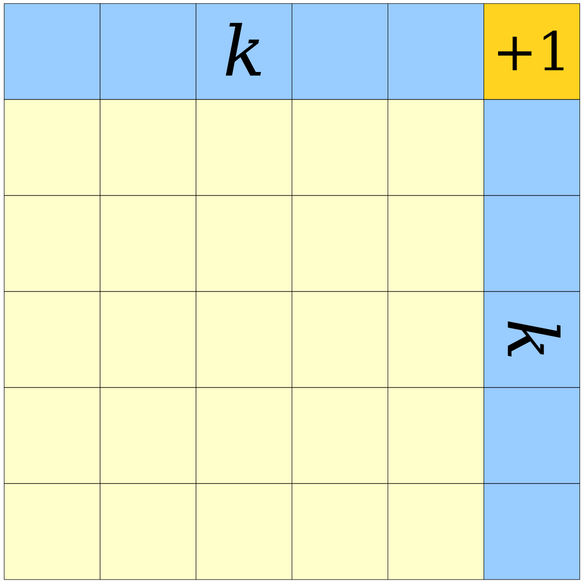
$$5 = 2 \cdot \mathbf{2} + 1 = \mathbf{3}^2 - \mathbf{2}^2$$

$$7 = 2 \cdot \mathbf{3} + 1 = \mathbf{4}^2 - \mathbf{3}^2$$

$$9 = 2 \cdot \mathbf{4} + 1 = \mathbf{5}^2 - \mathbf{4}^2$$

Theorem: For any odd integer n,
there exist integers r and s where $r^2 - s^2 = n$.

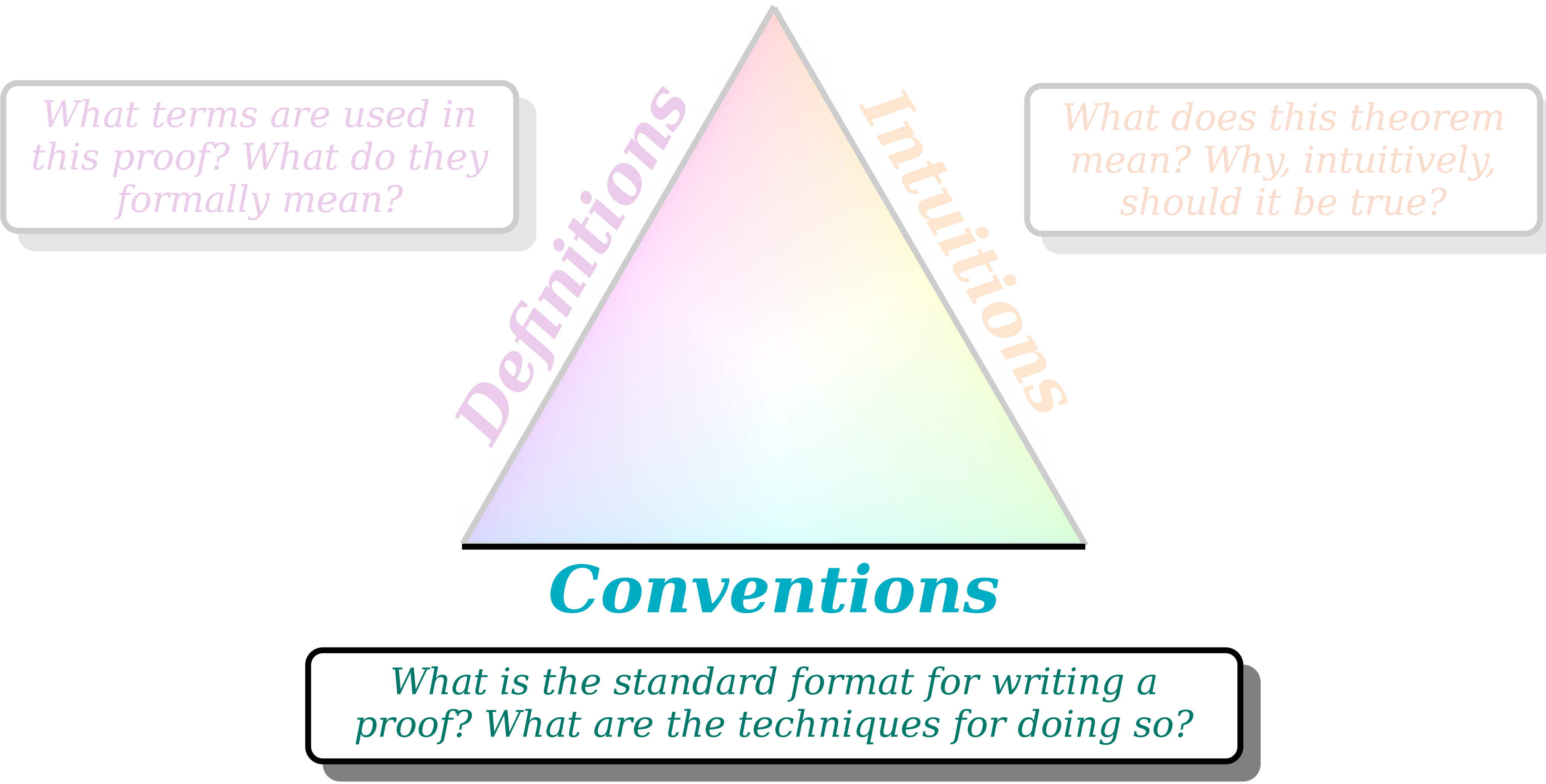
Let's Draw Some Pictures!



$$(k+1)^2 - k^2 = 2k+1$$

Theorem: For any odd integer n,
there exist integers r and s where $r^2 - s^2 = n$.

Universal and Existential Statements



Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

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Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$
\begin{aligned}
r^2 - s^2 &= (k+1)^2 - k^2 \\
&= k^2 + 2k + 1 - k^2 \\
&= 2k + 1 \\
&= n.
\end{aligned}
$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since … ere
$n = 2k$ … see
that

We ask the reader to make an arbitrary choice. Rather than specifying what **n** is, we're signaling to the reader that they could, in principle, supply any choice **n** that they'd like.

$$= 2k + 1$$

$$= n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k +$ … that

$$r^2 - s^2 = (k + \ldots$$

$$= k^2 \ldots$$

$$= 2k + 1$$

$$= n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

As always, it's helpful to write out what we need to demonstrate with the rest of the proof.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$
\begin{aligned}
r^2 - s^2 &= (k+1)\ldots \\
&= k^2 + 2\ldots \\
&= 2k + \ldots \\
&= n.
\end{aligned}
$$

This means that $r^2 - s^2 = n$, wh… to show. ■

We're trying to prove an existential statement. The easiest way to do that is to just give concrete choices of the objects being sought out.

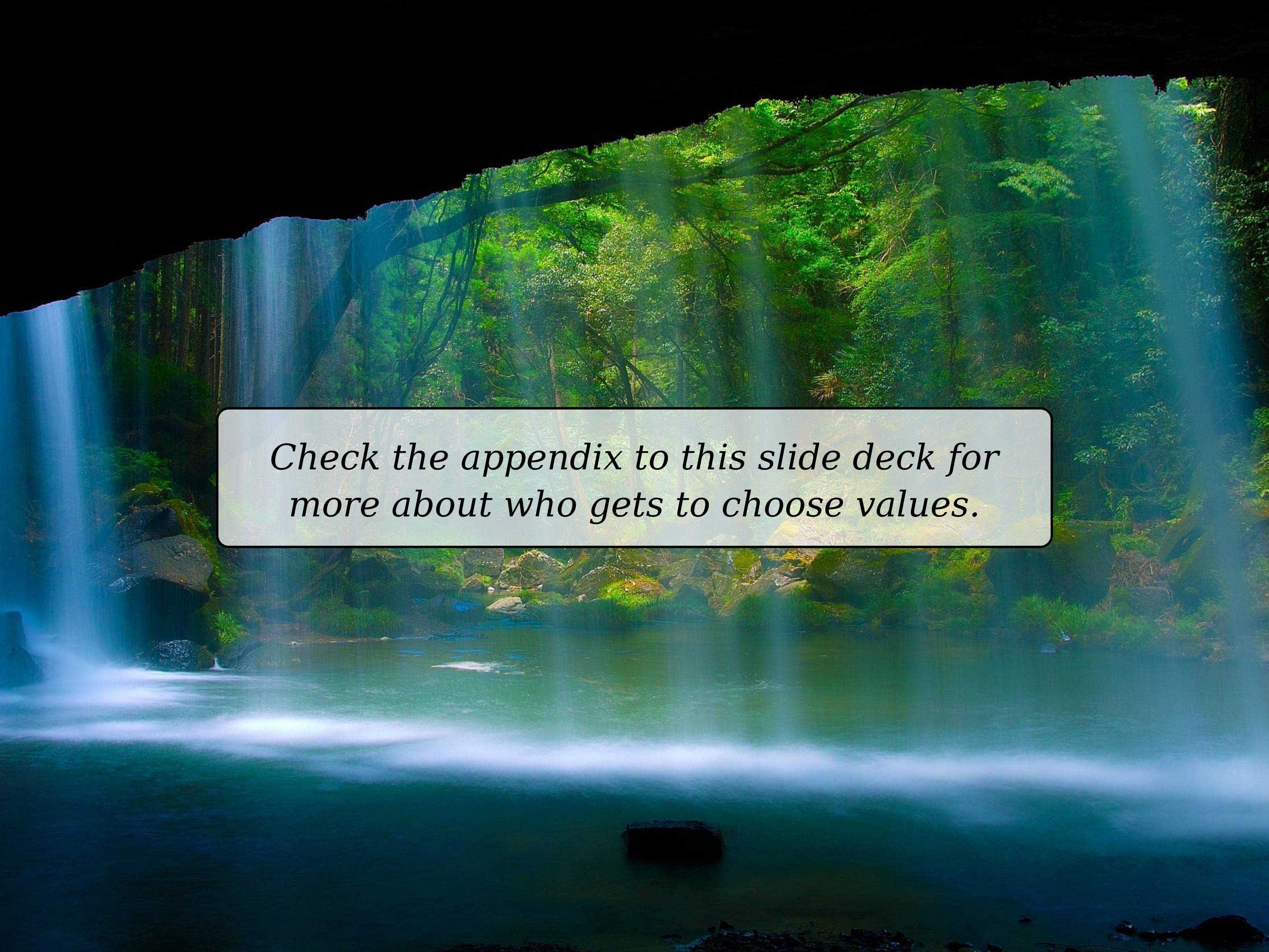
Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$
\begin{aligned}
r^2 - s^2 &= (k+1)^2 - k^2 \\
&= k^2 + 2k + 1 - k^2 \\
&= 2k + 1 \\
&= n.
\end{aligned}
$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■


Check the appendix to this slide deck for more about who gets to choose values.

Mathematical Proofs

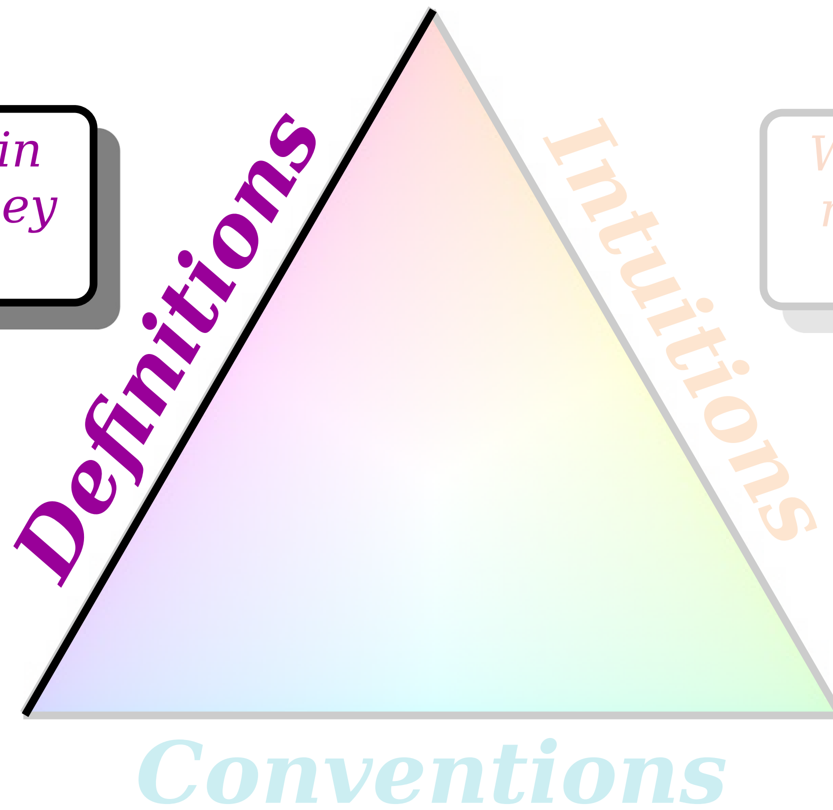
1. Announcements
2. What is a Proof? (Proof as Dialogue)
3. The Proof Writing Triad
4. Writing Our First Proof
5. Our Second Proof
6. Interlude: More Even/Odd Results
7. Universal and Existential Statements
8. **Floors, Ceilings, and Proofs by Cases**
9. Recap, Action Items, and What's Next?

Floors and Ceilings

Theorem: If n is an integer,
then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Floors and Ceilings

What terms are used in this proof? What do they formally mean?



What does this theorem mean? Why, intuitively, should it be true?

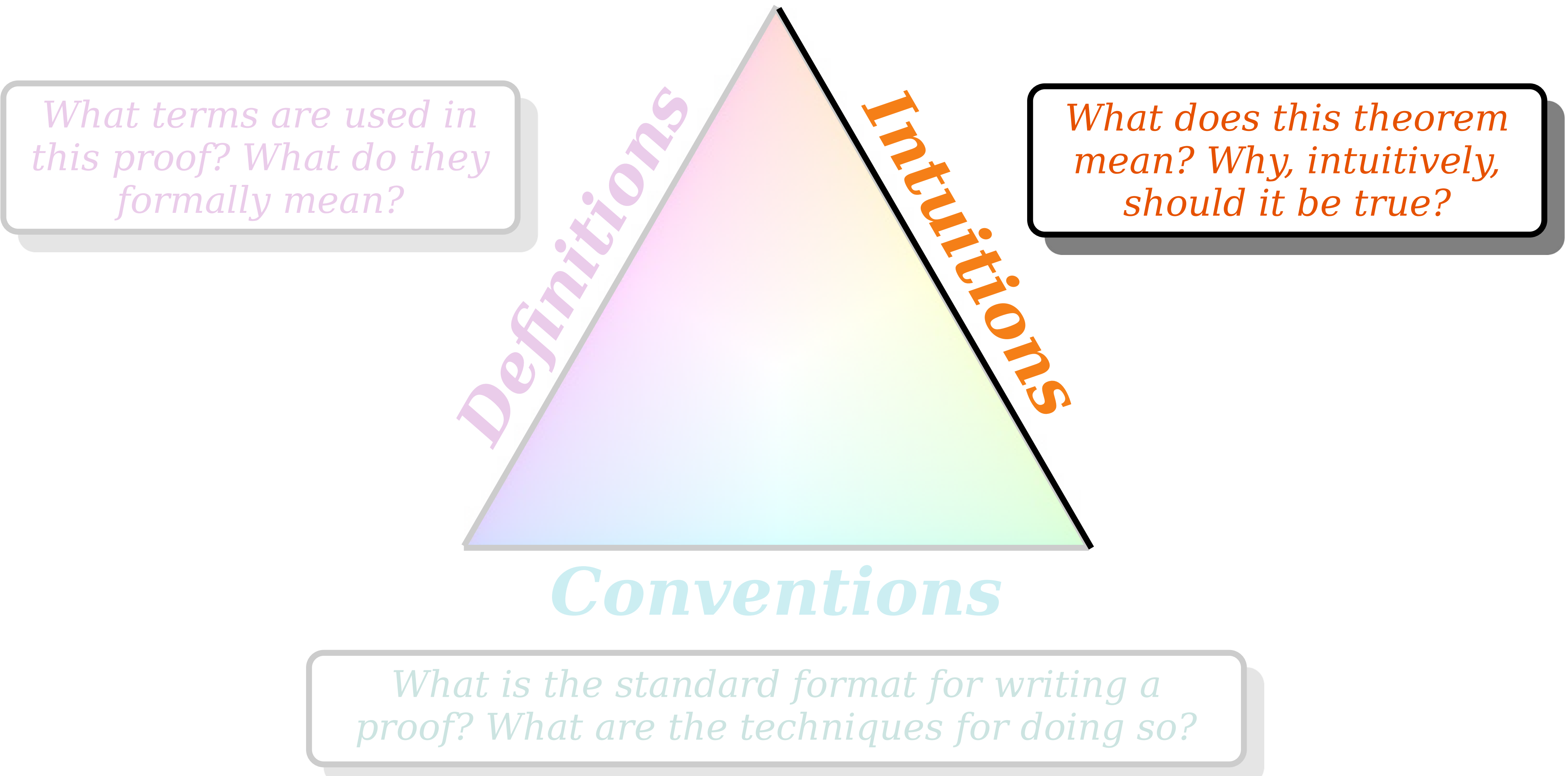
What is the standard format for writing a proof? What are the techniques for doing so?

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Floors and Ceilings

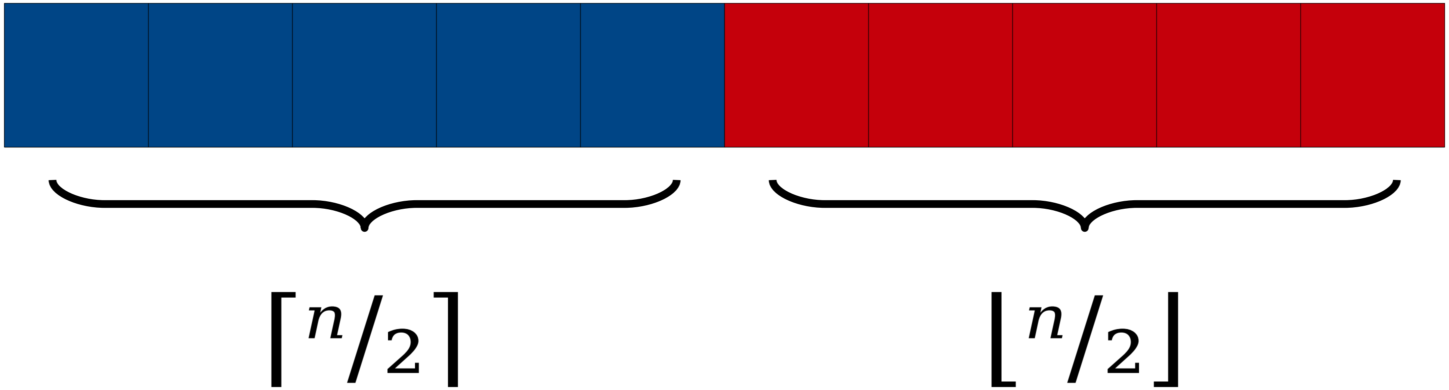
- The notation $\lceil x \rceil$ represents the ***ceiling*** of x, the smallest integer greater than or equal to x.
  - ***Intuition:*** Start at x on the number line, then move to the right while you're not on a tick mark.
  - What is $\lceil 1 \rceil$? What's $\lceil 1.2 \rceil$? What's $\lceil -1.2 \rceil$?
- The notation $\lfloor x \rfloor$ represents is the ***floor*** of x, the largest integer less than or equal to x.
  - ***Intuition:*** Start at x on the number line, then move to the left while you're not on a tick mark.
  - What is $\lfloor 1 \rfloor$? What's $\lfloor 1.2 \rfloor$? What's $\lfloor -1.2 \rfloor$?

Floors and Ceilings



Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

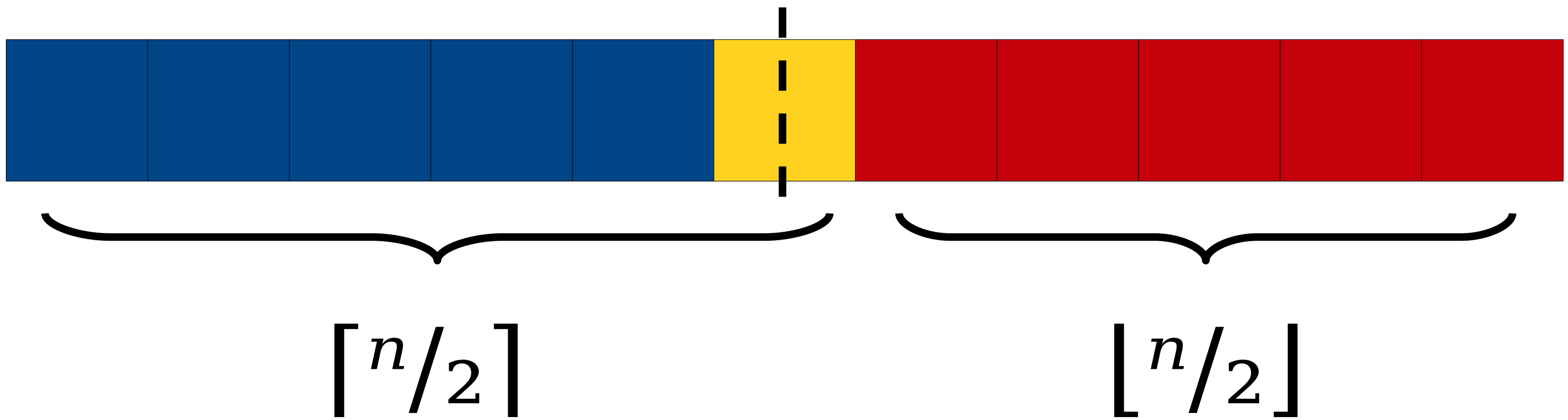
Let's Draw Some Pictures!



$$n = 2k$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Let’s Draw Some Pictures!



$$n = 2k + 1$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Floors and Ceilings

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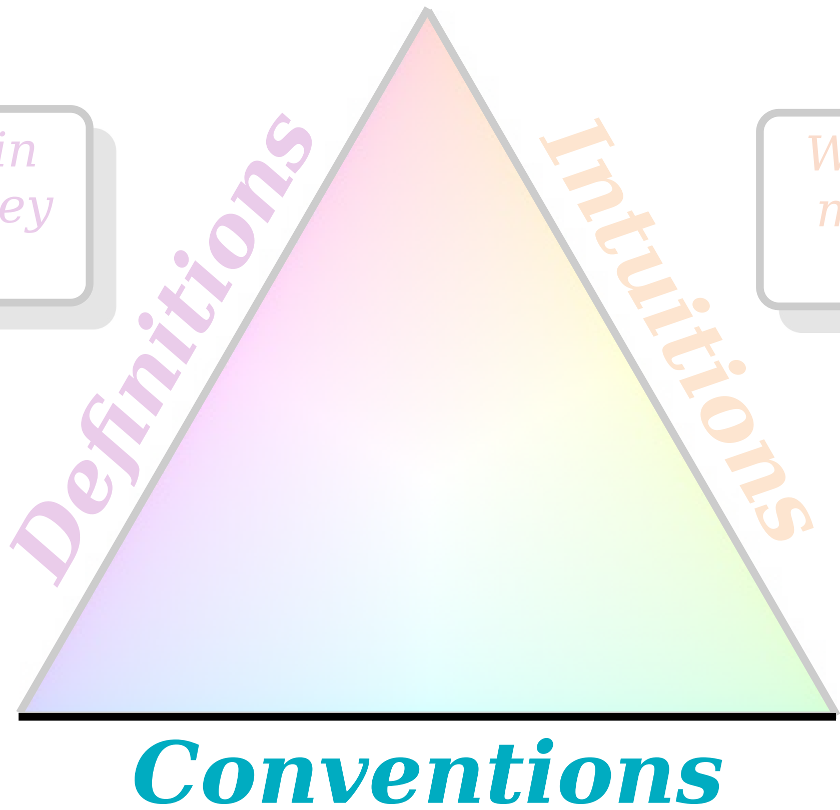

What does this theorem mean? Why, intuitively, should it be true?

What is the standard format for writing a proof? What are the techniques for doing so?

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even.

Case 2: n is odd.

This is called a *proof by cases* (or *proof by exhaustion*). We split apart into one or more cases and confirm that the result is indeed true in each of them.

(Think of it like an if/else or switch statement.)

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

$$\begin{aligned} \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor &= \left\lceil \frac{2k}{2} \right\rceil + \left\lfloor \frac{2k}{2} \right\rfloor \\ &= \lceil k \rceil + \lfloor k \rfloor \\ &= 2k \\ &= n. \end{aligned}$$

Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{2k+1}{2} \right\rceil + \left\lfloor \frac{2k+1}{2} \right\rfloor$$

> At the end of a split into cases, it's a nice courtesy to explain to the reader what it was that you established in each case.

$$= n.$$

In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

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Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

$$\begin{aligned} \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor &= \left\lceil \frac{2k+1}{2} \right\rceil + \left\lfloor \frac{2k+1}{2} \right\rfloor \\ &= \left\lceil k + \frac{1}{2} \right\rceil + \left\lfloor k + \frac{1}{2} \right\rfloor \\ &= (k+1) + k \\ &= 2k+1 \\ &= n\,. \end{aligned}$$

In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required. ■

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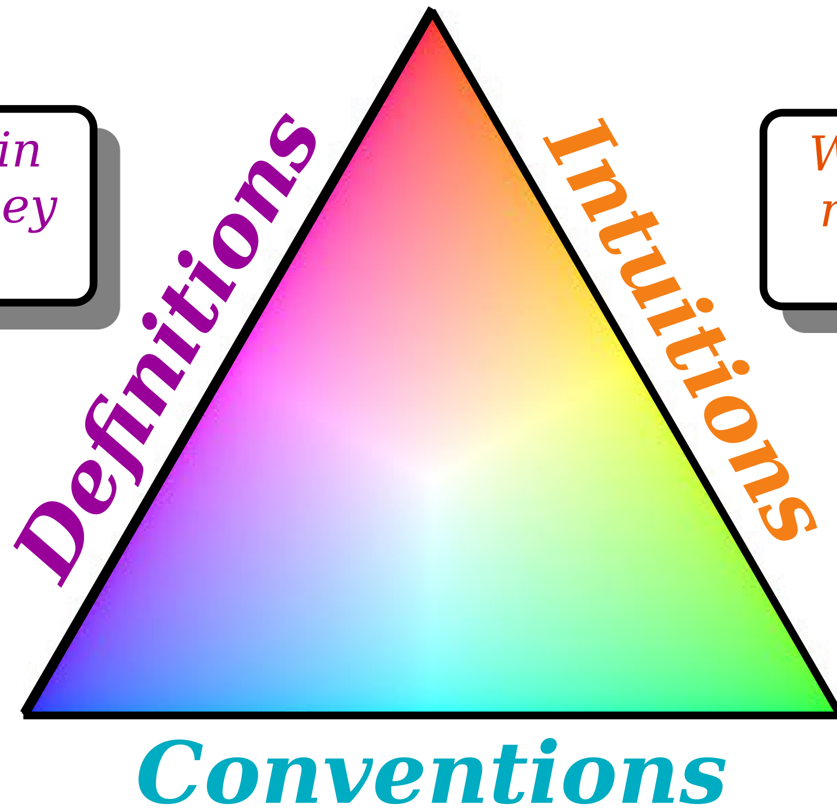
Recap

What terms are used in this proof? What do they formally mean?



What does this theorem mean? Why, intuitively, should it be true?

What is the standard format for writing a proof? What are the techniques for doing so?

Key Takeaway: Writing a good proof requires a blend of definitions, intuitions, and conventions.

Recap

- ***Definitions*** tell us what we need to do in a proof. Many proofs directly reference these definitions.

An integer n is ***even*** if there is an integer k where $n = 2k$.

An integer n is ***odd*** if there is an integer k where $n = 2k+1$.

Recap

- ***Definitions*** tell us what we need to do in a proof. Many proofs directly reference these definitions.
- Building ***intuition*** for results requires creativity, trial, and error.

Let's draw some pictures!

Let's do some math!

Let's try some examples!

Recap

- ***Definitions*** tell us what we need to do in a proof. Many proofs directly reference these definitions.
- Building ***intuition*** for results requires creativity, trial, and error.
- Mathematical proofs have established ***conventions*** that increase rigor and readability.

Recap

- Mathematical proofs have established ***conventions*** that increase rigor and readability.
  - Procedural:
    - Prove universal statements by making arbitrary choices.
    - Prove existential statements by making concrete choices.
    - Prove “If P, then Q” by assuming P and proving Q.
  - Stylistic:
    - Write in complete sentences.
    - Number sub-formulas when referring to them.
    - Summarize what was shown in proofs by cases.
    - Articulate your start and end points.

Your Action Items

- ***Read the following guides.***
  - Guide to ∈ and ⊆
  - Guide to Proofs
  - Guide to Partners.
- ***Finish and submit Problem Set 0.***
  - Don't put this off until the last minute!
- ***(Optional) Fill out the following forms.***
  - Problem Set Matchmaker (***link***)
  - CS103 ACE Application (***link***)

Next Time

- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does "if *P*, then *Q*" mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.

Appendix:

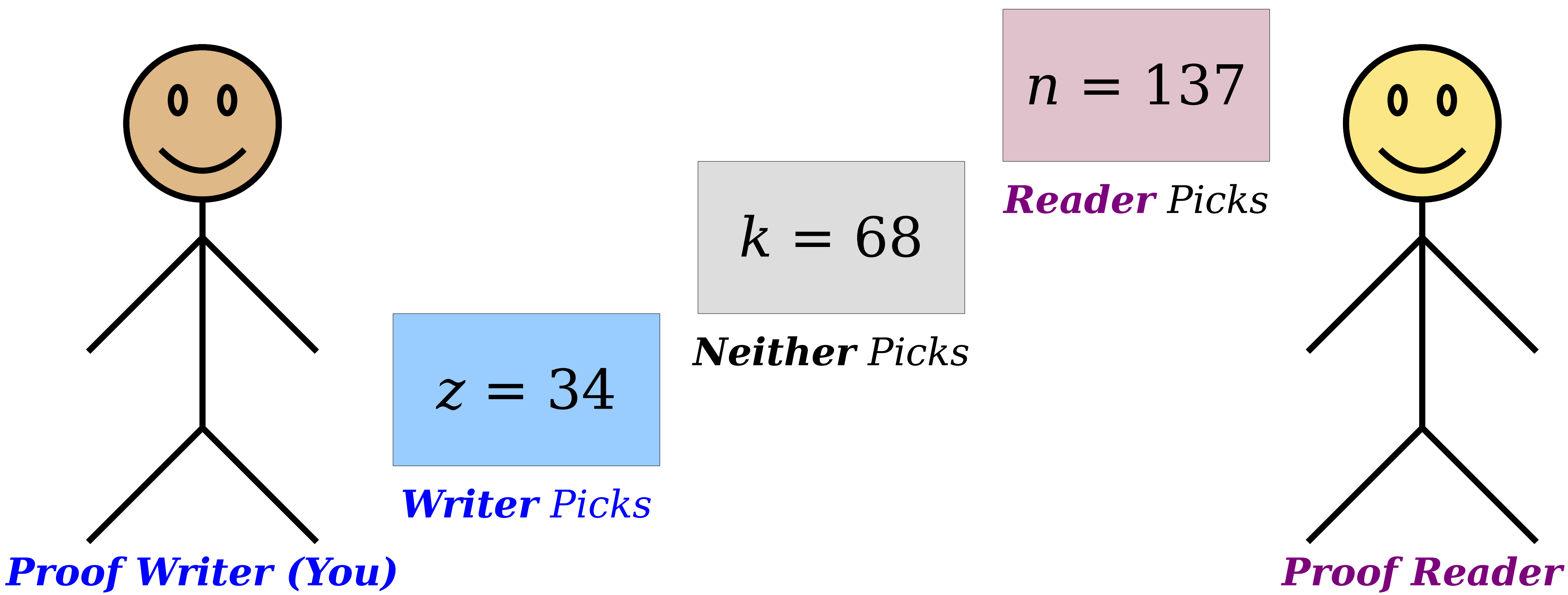
Proofs as a Dialogue

Proofs as a Dialogue

Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.

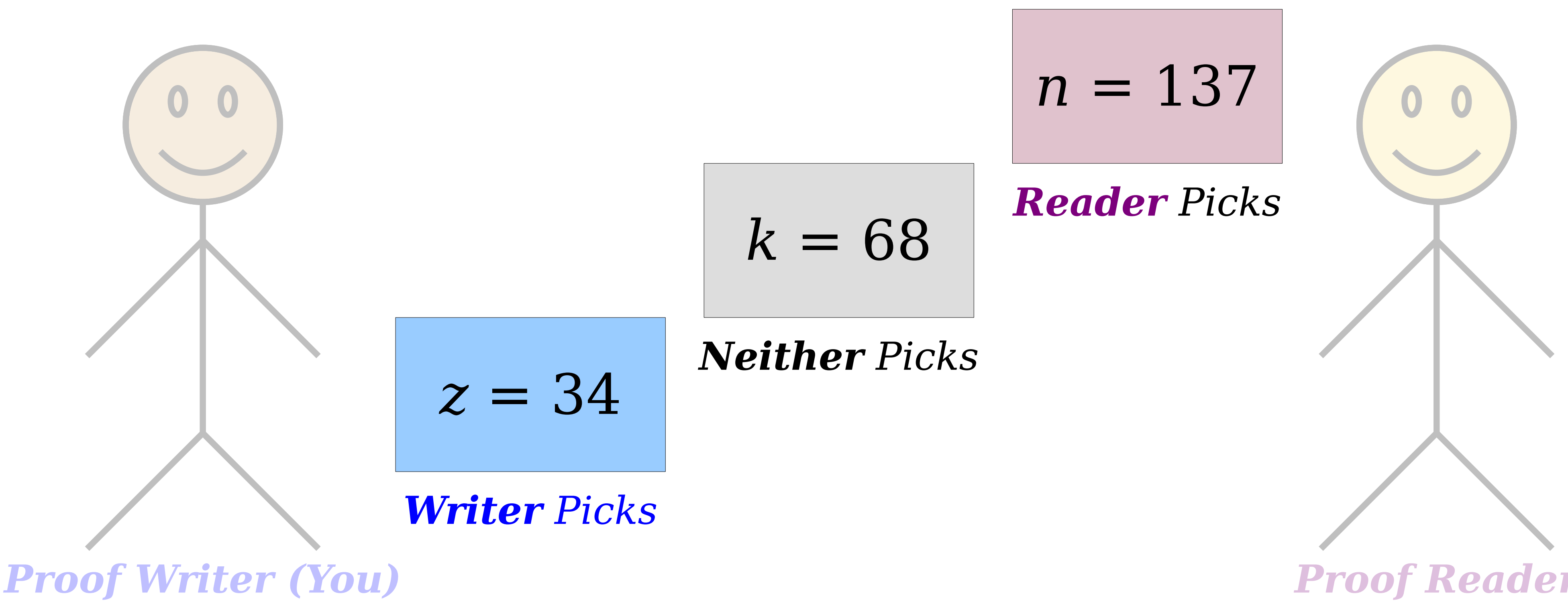


Each of these variables has a distinct, assigned value.

Each variable was either picked by the reader, picked by the writer, or has a value that can be determined from other variables.

Since …

Now, let $z = k - 34$.



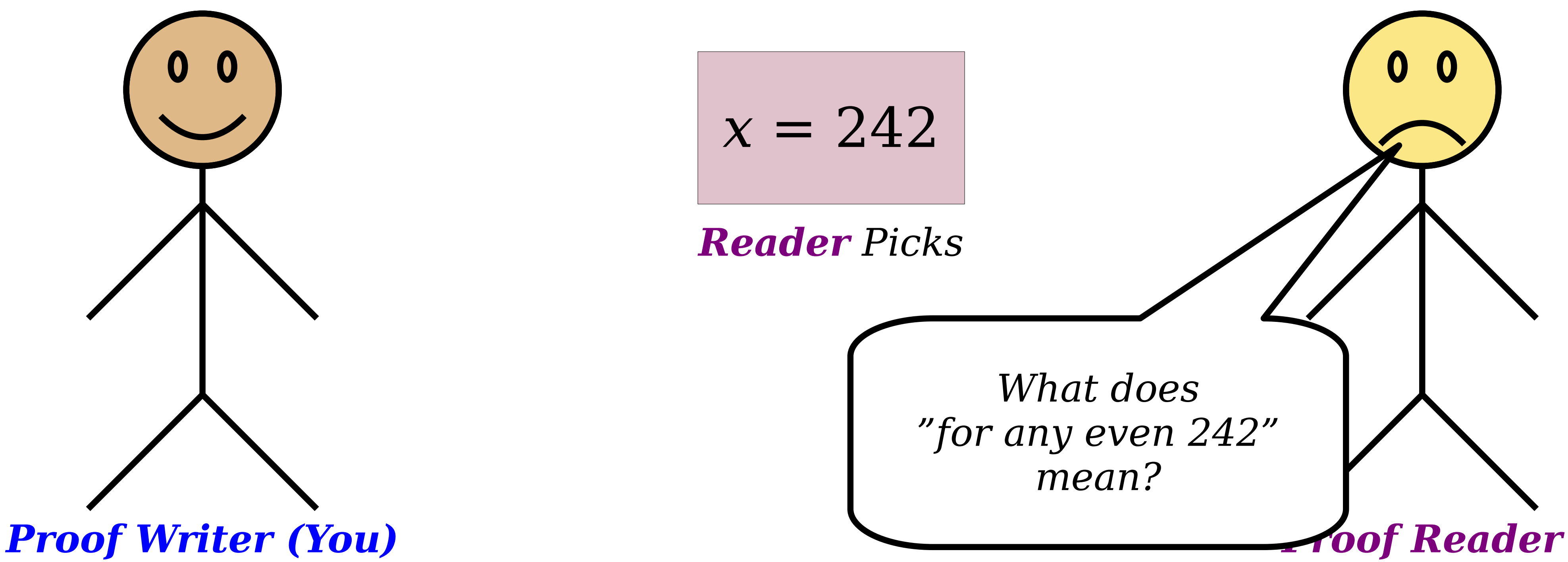
Who Owns What?

- The ***reader*** chooses and owns a value if you use wording like this:
  - Pick a natural number n.
  - Consider some $n \in \mathbb{N}$.
  - Fix a natural number n.
  - Let n be a natural number.
- The ***writer*** (you) chooses and owns a value if you use wording like this:
  - Let $r = n + 1$.
  - Pick $s = n$.
- ***Neither*** of you chooses a value if you use wording like this:
  - Since n is even, we know there is some $k \in \mathbb{Z}$ where $n = 2k$.
  - Because n is odd, there must be some integer k where $n = 2k + 1$.

Proofs as a Dialogue

Let x be an arbitrary even integer.

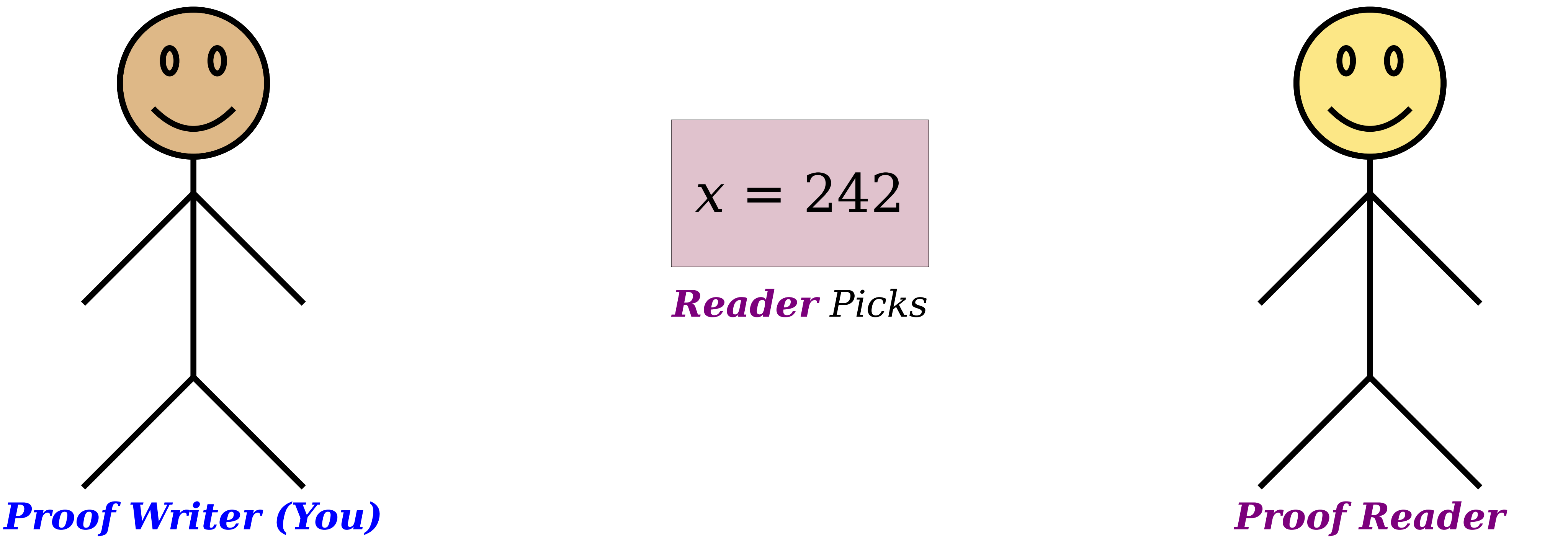
Then for any even x, we know that $x+1$ is odd.



Proofs as a Dialogue

Let x be an arbitrary even integer.

Since x is even, we know that $x+1$ is odd.



Every variable needs a value.

Avoid talking about "all x" or "every x" when manipulating something concrete.

To prove something is true for any choice of a value for x, let the reader pick x.

Once you've said something like

Let x be an integer.
Consider an arbitrary $x \in \mathbb{Z}$.
Pick any x.

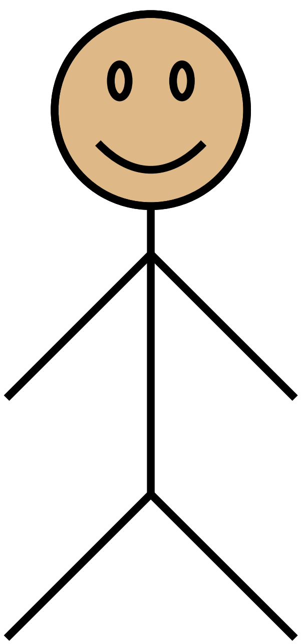
Do not say things like the following:

This means that ***for any*** $x \in \mathbb{Z}$ ...
So ***for all*** $x \in \mathbb{Z}$ ...

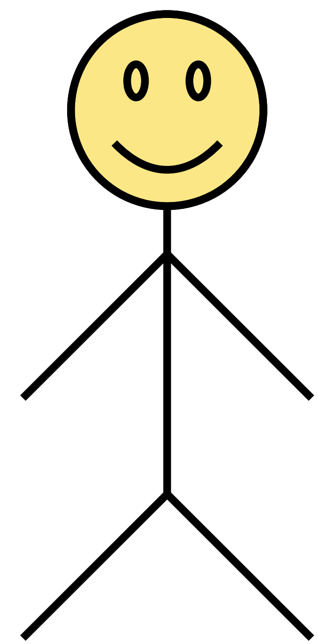
Proofs as a Dialogue

Pick two integers m and n where $m+n$ is odd.

Let $n = 1$, which means that $m+1$ is odd.



Proof Writer (You)



Proof Reader

Proofs as a Dialogue

Pick two integers m and n where $m+n$ is odd.

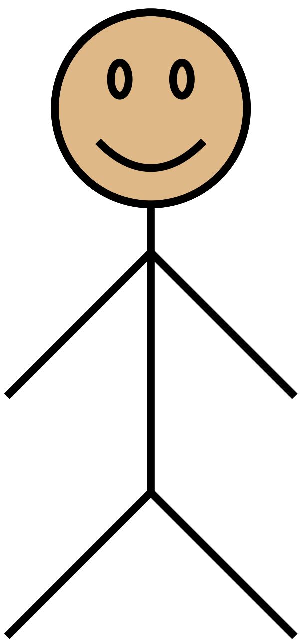
Let $n = 1$, which means that $m+1$ is odd.

$m = 103$

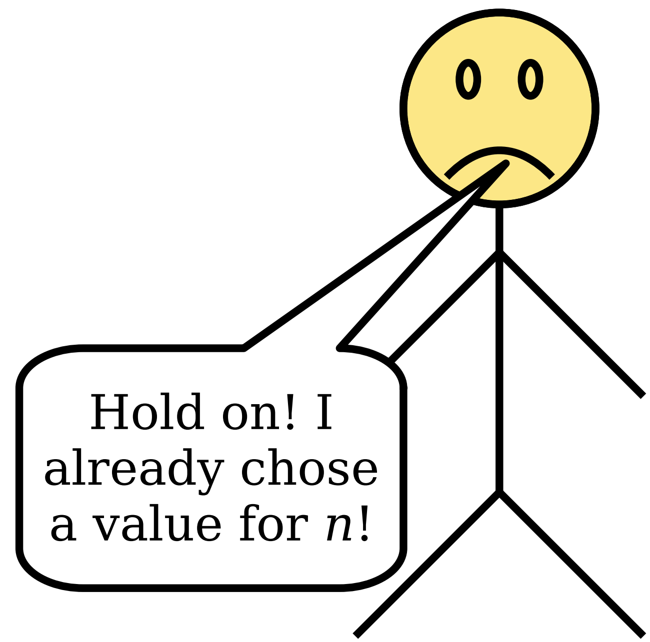
Reader *Picks*

$n = 166$

Reader *Picks*



Proof Writer (You)



Proof Reader

Proofs as a Dialogue

Let $n = 1$.

Pick any integer m where $m+1$ is odd.

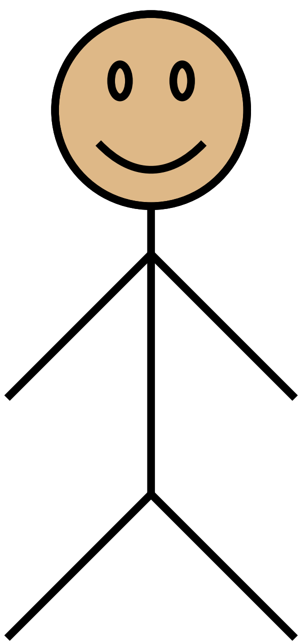
Do we even need n here?

$m = 166$

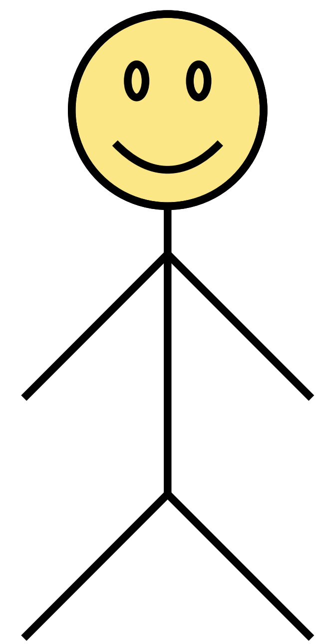
Reader *Picks*

$n = 1$

Writer *Picks*



Proof Writer (You)



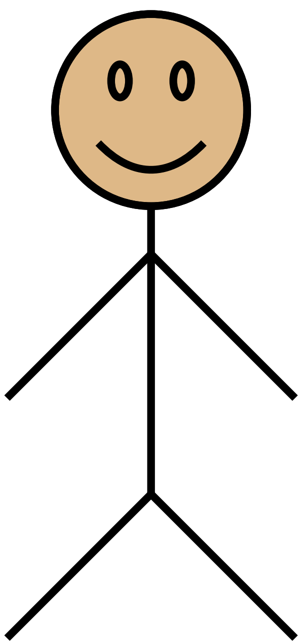
Proof Reader

Proofs as a Dialogue

Pick any integer m where $m+1$ is odd.

$m = 166$

***Reader** Picks*



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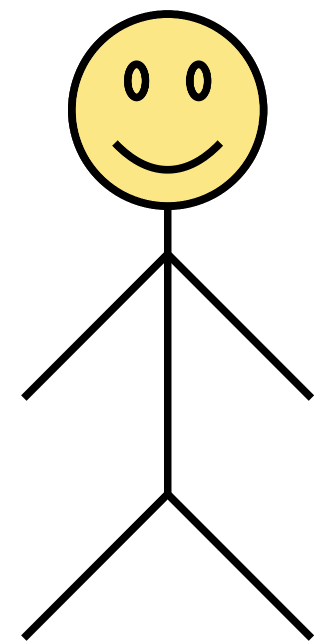

Proof Reader

Be mindful of who owns what variable.

Don’t change something you don’t own.

You don’t always need to name things, especially if they already have a name.